C

GEOMETRIC PROPERTIES OF PLANE AREAS

C.1 FIRST MOMENTS OF AREA; CENTROID

Definitions. The solutions of most problems in this book involve one or more geometric properties of plane areas[4]—area, centroid, second moment, etc. The total *area* of a plane surface enclosed by bounding curve B is defined by the integral

$$A = \int_A dA \tag{C-1}$$

which is understood to mean a summation of differential areas dA over two spatial variables, such as y and z in Fig. C-1.

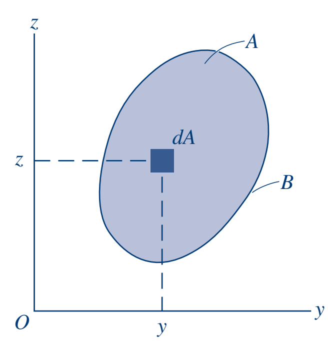


FIGURE C-1 A plane area.

The *first moments* of the area A about the y and z axes, respectively, are defined as

$$Q_y = \int_A z dA, \quad Q_z = \int_A y dA \tag{C-2}$$

Q_y and Q_z are called first moments because the distances z and y appear to the first power in the defining integrals.

The *centroid* of an area is its "geometric center." The coordinates $(\bar{y}, \bar{z})$ of the centroid C (Fig. C-2) are defined by the first-moment equations

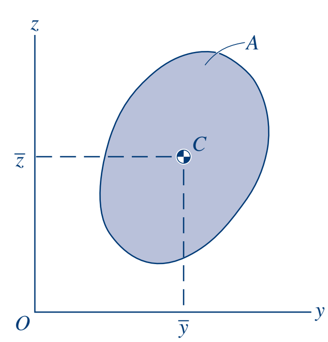


FIGURE C-2 Location of the centroid of an area.

$$\bar{y}A = \int_A y dA, \quad \bar{z}A = \int_A z dA \tag{C-3}$$

For simple geometric shapes (e.g., rectangles, triangles, circles) there are closed-form formulas for the geometric properties of plane areas. A number of these are

[4]The word *area* is used in two senses: In one sense, the word refers to the portion of a plane surface that lies within a prescribed bounding curve, like the area bounded by the closed curve B in Fig. C-1; in the second sense, the word refers to the quantity of surface within the bounding curve [Eq. (C-1)].

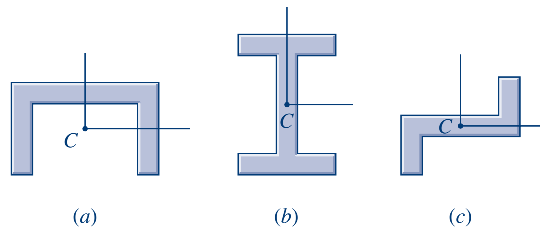


FIGURE C-3 Three types of area symmetry.

given in a table inside the back cover of this book.[5] An area may possess one of the three symmetry properties illustrated in Fig. C-3. If an area has *one axis of symmetry,* like the vertical axis of the C-section in Fig. C-3*a*, the centroid of the area lies on that axis. If the area has *two axes of symmetry,* like the wide-flange shape in Fig. C-3*b*, then the centroid lies at the intersection of those axes. Finally, if the area is *symmetric about a point,* like the Z-section in Fig. C-3*c*, the center of symmetry is the centroid of the area.

Composite Areas. Many structural shapes are composed of several parts, each of which is a simple geometric shape. For example, each of the areas in Fig. C-3 can be treated as a composite area made up of three rectangular areas. Since the integrals in Eqs. (C-1) through (C-3) represent summations over the total area A, they can be evaluated by summing the contributions of the constituent areas A_i, giving

$$A = \sum_i A_i, \quad Q_z = \bar{y}A = \sum_i \bar{y}_i A_i, \quad Q_y = \bar{z}A = \sum_i \bar{z}_i A_i \qquad \text{(C-4)}$$

Note that $\bar{y}$ in Fig. C-4 can be determined directly from the symmetry of the figure about the z' axis.

COMPOSITE-AREA PROCEDURE FOR LOCATING THE CENTROID

1. Divide the composite area into simpler areas for which there exist formulas for area and for the coordinates of the centroid. (See the table inside the back cover.)
2. Establish a convenient set of reference axes (y, z).
3. Determine the area, A, using Eq. (C-4a).
4. Calculate the coordinates of the composite centroid, $(\bar{y}, \bar{z})$, using Eqs. (C-4b, c).

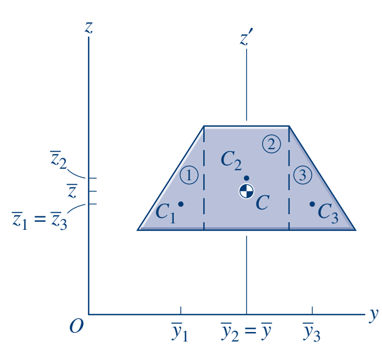


FIGURE C-4 A composite area.

[5]The reader may consult textbooks on integral calculus or statics for exercises in evaluating the integrals in Eqs. C-1 through C-3 for specific shapes.

EXAMPLE C-1

Locate the centroid of the L-shaped area in Fig. 1.

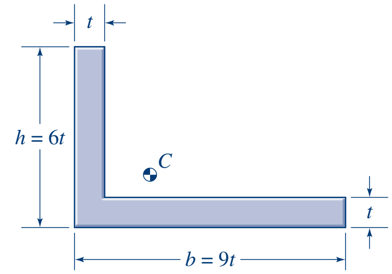


Fig. 1

Solution A—Addition Method Following the procedure outlined above, we divide the L-shaped area into two rectangles, as shown in Fig. 2. The y and z axes are located along the outer edges of the area, with the origin at the lower-left corner. Since the composite area consists of only two areas, the composite centroid, C, lies between C_1 and C_2 on the line joining the two centroids, as illustrated in Fig. 2.

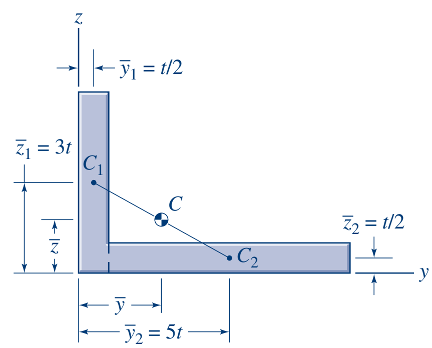


Fig. 2

Area: From Eq. (C-4a),

$$A = A_1 + A_2 = (6t)(t) + (8t)(t) = 14t^2 \qquad (1)$$

Centroid: From Eqs. (C-4b) and (C-4c),

$$\bar{y}A = \bar{y}_1 A_1 + \bar{y}_2 A_2 = (t/2)(6t^2) + 5t(8t^2) = 43t^2$$

$$\bar{y} = \frac{43t^3}{14t^2} = \frac{43}{14}t = 3.07t \qquad \textbf{Ans.} \qquad (2)$$

$$\bar{z}A = \bar{z}_1 A_1 + \bar{z}_2 A_2 = (3t)(6t^2) + (t/2)(8t^2) = 22t^3$$

$$\bar{z} = \frac{22t^3}{14t^2} = \frac{22}{14}t = 1.57t \qquad \textbf{Ans.} \qquad (3)$$

Solution B—Subtraction Method Sometimes (although not in this particular example) it is easier to solve composite-area problems by treating the area as the net area obtained by subtracting one or more areas from a larger area. Then, in Eqs. (C-4), the A_i's of the removed areas are simply taken as negative areas. This method will now be applied to the L-shaped area in Fig. 1 by treating it as a larger rectangle from which a smaller rectangle is to be subtracted (Fig. 3). Area A_1 is the large rectangle $PQRS$; area A_2 is the smaller unshaded rectangle. The composite centroid, C, lies along the line joining the two centroids, C_1 and C_2, but it does not fall between them.

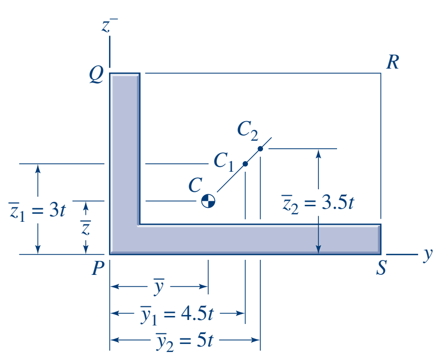


Fig. 3

Area: From Eq. (C-4a),

$$A = A_1 + A_2 = (9t)(6t) + [-(8t)(5t)] = 14t^2 \quad (4)$$

Centroid: From Eqs. (C-4b) and (C-4c),

$$\bar{y}A = \bar{y}_1 A_1 + \bar{y}_2 A_2 = (4.5t)(54t^2) + [(5t)(-40t^2)] = 43t^3$$

$$\bar{y} = 43t^3/14t^2 = 3.07t \qquad \textbf{Ans.} \quad (5)$$

$$\bar{z}A = \bar{z}_1 A_1 + \bar{z}_2 A_2 = (3t)(54t^2) + [(3.5t)(-40t^2)] = 22t^3$$

$$\bar{z} = 22t^3/14t^2 = 1.57t \qquad \textbf{Ans.} \quad (6)$$

C.2 MOMENTS OF INERTIA OF AN AREA

Definitions of Moments of Inertia. The *moments of inertia* of a plane area (Fig. C-5) about axes y and z in the plane are defined by the integrals

$$I_y = \int_A z^2 dA, \quad I_z = \int_A y^2 dA \quad \text{(C-5)}$$

These are called the *moment of inertia with respect to the y axis* and the *moment of inertia with respect to the z axis,* respectively. Since each integral involves the square of the distance of the elemental area dA from the axis involved, these quantities are called *second moments of area.* These moments of inertia appear primarily in formulas for bending of beams (see Chapter 6).

The moments of inertia defined in Eqs. (C-5) are with respect to axes that lie in the plane of the area under consideration. The second moment of area about the x axis, that is, with respect to the origin O, is called the *polar moment of inertia* of the area. It is defined by

$$I_p = \int_A \rho^2 dA \quad \text{(C-6)}$$

Since, by the Pythagorean theorem, $\rho^2 = y^2 + z^2$, I_p is related to I_y and I_z by

$$I_p = I_y + I_z \quad \text{(C-7)}$$

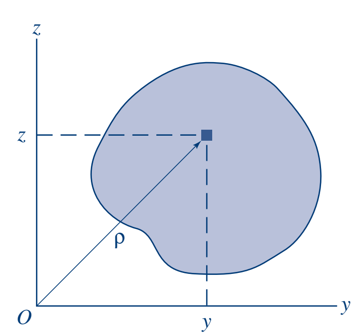


FIGURE C-5 A plane area.

Since Eqs. (C-5) and (C-6) involve squares of distances, I_y, I_z, and I_p are always positive. All have the dimension of $(\text{length})^4$ − in^4, mm^4, etc.

A table listing formulas for coordinates of the centroid and for moments of inertia of a variety of shapes may be found inside the back cover of this book. The most useful formulas for moments of inertia and for polar moment of inertia are derived here.

Moments of Inertia of a Rectangle: For the rectangle in Fig. C-6*a*, Eq. (C-5a) gives

$$I_y = \int_A z^2 dA = \int_{-h/2}^{h/2} z^2 (b\,dz) = b \left.\frac{z^3}{3}\right|_{-h/2}^{h/2} = \frac{bh^3}{12}$$

where the *y* axis passes through the centroid and is parallel to the two sides of length *b*. I_z may be derived in an analogous manner, so the moments of inertia of a rectangle for the two centroidal axes parallel to the sides of the rectangle are:

$$I_y = \frac{bh^3}{12}, \quad I_z = \frac{hb^3}{12} \tag{C-8}$$

Polar Moment of Inertia of a Circle about its Center: Letting $dA = 2\pi\rho d\rho$, the area of the dark-shaded ring in Fig. C-6*b*, and using Eq. (C-6), we can determine the polar moment of inertia of a circle about its center:

$$I_p = \int_A \rho^2 dA = \int_0^r \rho^2 (2\pi\rho d\rho) = \frac{\pi r^4}{2}$$

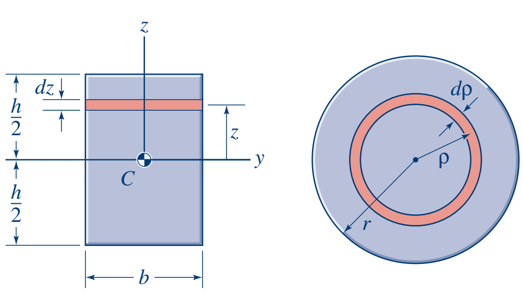
$$I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \tag{C-9}$$

Radii of Gyration. A length called the *radius of gyration* is defined for each moment of inertia by the formulas

$$r_y = \sqrt{\frac{I_y}{A}}, \quad r_z = \sqrt{\frac{I_z}{A}} \tag{C-10}$$

These lengths are used to simplify several formulas in Chapters 6 and 10. If these formulas are written in the form

$$I = Ar^2$$



(*a*) A rectangular area. (*b*) A circular area.

FIGURE C-6 Notation for calculating moments of inertia and polar moment of inertia.

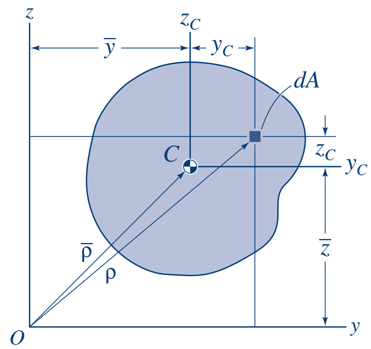


FIGURE C-7 Notation for deriving parallel-axis theorems.

then it is clear that the radius of gyration is the distance at which the entire area could be concentrated and still give the same value I, of moment of inertia about a given axis.

Parallel-Axis Theorems for Moments of Inertia. Let the (y, z) pair of axes be parallel to centroidal axes (y_C, z_C), as shown in Fig. C-7. The centroid, C, is located with respect to the (y, z) axes by the centroidal coordinates $(\bar{y}, \bar{z})$. Since, from Fig. C-7, $z = \bar{z} + z_C$, the moment of inertia I_y is given by

$$I_y = \int_A z^2 dA = \int_A (\bar{z} + z_C)^2 dA$$

$$= \bar{z}^2 A + 2\bar{z}\int_A z_C dA + \int_A z_C^2 dA$$

$$\boxed{I_y = \bar{z}^2 A + I_{y_C}} \tag{C-11a}$$

The term $\int_A z_C dA$ vanishes since the y_C axis passes through the centroid; the term I_{y_C} is the *centroidal moment of inertia* about the y_C axis. The $\bar{z}^2 A$ term is the moment of inertia that area A would have about the y axis if all of the area were to be concentrated at the centroid. Since this term is always zero or positive, the centroidal moment of inertia is the minimum moment of inertia with respect to all parallel axes.

By the same procedure that was used to obtain Eq. (C-11a), we get

$$\boxed{I_z = \bar{y}^2 A + I_{z_C}} \tag{C-11b}$$

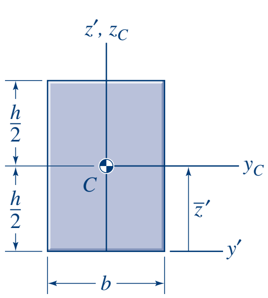


FIGURE C-8 A rectangular area with two sets of axes.

Equations C-11 are called *parallel-axis theorem for moments of inertia.*

As a simple example of calculations based on the parallel-axis theorem, let us determine the moment of inertia of the rectangle in Fig. C-8 about the y' axis along an edge of length b. From Eq. (C-11a),

$$I_{y'} = (\bar{z}')^2 A + I_{y_C} = \left(\frac{h}{2}\right)^2 (bh) + \frac{bh^3}{12} = \frac{bh^3}{3} \tag{C-12}$$

In a similar manner, a *parallel-axis theorem for the polar moment of inertia* may be derived. From Eq. (C-6) and Fig. C-7, the polar moment of inertia about point O is

$$I_{p_O} = \int_A \rho^2 dA = \int_A (y^2 + z^2) dA$$
$$= \int_A [(\overline{y} + y_C)^2 + (\overline{z} + z_C)^2] dA$$
$$= \int_A (\overline{y}^2 + 2\overline{y}y_C + y_C^2 + \overline{z}^2 + 2\overline{z}z_C + z_C^2) dA$$

$$\boxed{I_{p_O} = \overline{\rho}^2 A + I_{p_C}} \qquad \text{(C-13)}$$

since $\overline{y}^2 + \overline{z}^2 = \overline{\rho}^2$, $\int_A y_C dA = \int_A z_C dA = 0$, and $\int_A (y_C^2 + z_C^2) dA = I_{p_C}$. Note that Eq. (C-13) follows easily from Eq. (C-7) and Eqs. (C-11).

Moments of Inertia of Composite Areas. The moments of inertia of a composite area, like the one in Fig. C-4, may be computed by summing the contributions of the individual areas:

$$\boxed{I_y = \sum_i (A_y)_i, \quad I_z = \sum_i (I_z)_i} \qquad \text{(C-14)}$$

As an efficient procedure for calculating moments of inertia of composite areas, the following is suggested.

COMPOSITE-AREA PROCEDURE FOR CALCULATING SECOND MOMENTS

1. Divide the composite area into simpler areas for which there exist formulas for centroidal coordinates and moments of inertia. (See the table inside the back cover.)
2. Locate the centroid of each constituent area and establish centroidal reference axes (y_C, z_C) parallel to the given (y, z) axes.
3. Employ Eqs. (C-11) to compute the moments of inertia of the constituent areas with respect to the (y, z) axes and Eq. (C-14) to sum them.

The next example problem illustrates this procedure.

EXAMPLE C-2

Determine the centroidal moment of inertia I_y for the L-shaped section in Example C-1. (Here, in Fig. 1, the origin of the (y, z) reference frame is at the centroid of the composite area. The centroidal reference axes for the rectangular "legs" of the L-shaped area are (y_1, z_1) and (y_2, z_2), respectively.)

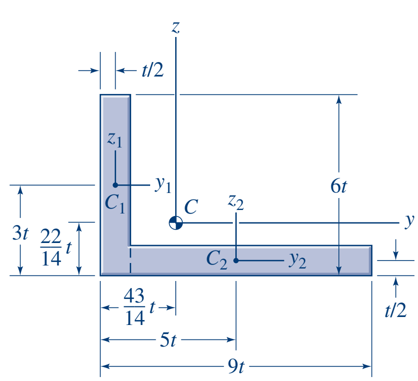


Fig. 1

Solution We can combine Eqs. (C-14) with the parallel axis theorems, Eqs. (C-11), to compute the required moments of inertia.

$$I_y = (I_y)_1 + (I_y)_2 = [(I_{y_C})_1 + A_1\overline{z}_1^2] + [(I_{y_C})_2 + A_2\overline{z}_2^2] \qquad (1)$$

where $(I_{y_c})_i$ is the moment of inertia of area A_i about its own centroidal y axis, and $\overline{z}_i$ is the z-coordinate of the centroid C_i measured in the $(y,$

z) reference frame with origin at the composite centroid, C. Referring to Fig. 1, we get

$$
\begin{aligned}
I_y &= (I_y)_1 + (I_y)_2 \\
&= \left[\frac{1}{12}(t)(6t)^3 + (t)(6t)\left(3t - \frac{22}{14}t\right)^2\right] \\
&+ \left[\frac{1}{12}(8t)(t)^3 + (t)(8t)\left(\frac{1}{2}t - \frac{22}{14}t\right)^2\right] \\
&= 18t^4 + \frac{600}{49}t^4 + \frac{2}{3}t^4 + \frac{450}{49}t^4 \\
&= \frac{842}{21}t^4 = 40.1t^4
\end{aligned}
$$

Ans. (2)

C.3 PRODUCT OF INERTIA OF AN AREA

Definition of Product of Inertia. Another geometric property of plane areas is called the *product of inertia,* which is defined by (refer to Fig. C-1)

$$I_{yz} = \int_A yz dA \tag{C-15}$$

The product of inertia is required in the study of bending of unsymmetric beams (Section 6.6).

As an example, let us determine the product of inertia of a rectangular area with respect to two sets of axes (Fig. C-9).

From Eq. (C-15) and Fig. C-9(a),

$$I_{yz} = \int_A yz\, dA = \int_0^h \int_0^b yz\, dy\, dz = \frac{b^2h^2}{4} \tag{C-16}$$

Now consider the product of inertia with respect to the (y', z') axes in Fig. C-9*b*. The y' axis is an axis of symmetry, and it passes through the centroid C. As

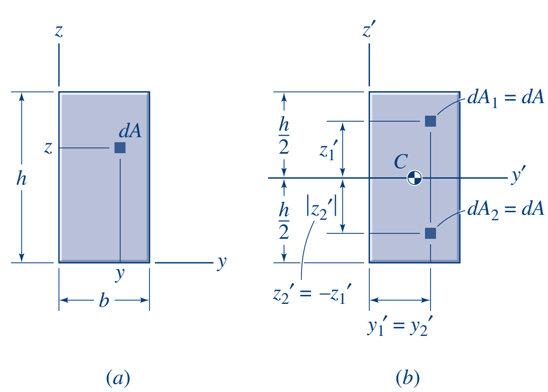


FIGURE C-9 A rectangular area with two sets of axes.

is clear from Fig. C-9*b*, when either reference axis is an axis of symmetry of the area, like the y' axis in this figure, the product of inertia is zero, since

$$I_{y'z'} = \int_A y'z'\,dA$$

and, because of symmetry (since $y_2' = y_1'$, but $z_2' = -z_1'$), the contributions of dA_1 and dA_2 to the integral cancel each other. Therefore,

$$I_{yz} = 0 \tag{C-17}$$

if either the y axis or the z axis is an axis of symmetry of the area.

Parallel-Axis Theorem for Product of Inertia of an Area. The procedure used to derive parallel-axis theorems for moments of inertia, leading to Eqs. (C-11) and (C-13), may be applied to derive a parallel-axis theorem for products of inertia. From Eq. (C-15) and Fig. C-7,

$$\begin{aligned} I_{yz} &= \int_A yz\,dA = \int_A (\bar{y} + y_C)(\bar{z} + z_C)\,dA \\ &= \bar{y}\bar{z}A + \bar{y}\int_A z_C\,dA + \bar{z}\int_A y_C\,dA + I_{y_Cz_C} \end{aligned}$$

Therefore, since y_C and z_C are coordinates in a centroidal reference frame, the *parallel-axis theorem for products of inertia of an area* is

$$\boxed{I_{yz} = \bar{y}\bar{z}A + I_{y_Cz_C}} \tag{C-18}$$

Just as for the moments of inertia, I_{yz} has one term that represents the product of inertia of an area A concentrated at the centroid, plus a *centroidal product of inertia* $I_{y_Cz_C}$.

Product of Inertia for Composite Areas. The summations for moments of inertia in Eqs. (C-14) are readily extended to the product of inertia of an area composed of several constituent areas:

$$\boxed{I_{yz} = \sum_i (I_{yz})_i} \tag{C-19}$$

EXAMPLE C-3

For the L-shaped area in Example C-2, use the composite-area procedure to determine the centroidal product of inertia, I_{yz}. (Note: Here the (y, z) reference frame is a centroidal reference frame for the whole area; (y_1, z_1) and (y_2, z_2) are centroidal reference frames for the constituent areas A_1 and A_2, respectively.) The centroid product of inertia relative to the (y, z) axes is given by

$$I_{yz} = (I_{y_1z_1} + A_1\bar{y}_1\bar{z}_1) + (I_{y_2z_2} + A_2\bar{y}_2\bar{z}_2)$$

It is very important to note that $\bar{y}_1$, $\bar{z}_1$, etc., are signed values, that is, some of them could be negative. By Eq. (C-17), $I_{y_1z_1} = I_{y_2z_2} = 0$. Therefore,

$$I_{yz} = A_1\bar{y}_1\bar{z}_1 + A_2\bar{y}_2\bar{z}_2$$

$$= (6t^2)\left(-\frac{43}{14}t + \frac{1}{2}t\right)\left(3t - \frac{22}{14}t\right)$$

$$+ (8t^2)\left(5t - \frac{43}{14}t\right)\left(-\frac{22}{14}t + \frac{1}{2}t\right)$$

or

$$I_{yz} = -\frac{270}{7}t^4 \qquad \textbf{Ans.}$$

Note that, since the centroid C_1 lies in the second quandrant and C_2 lies in the fourth quadrant, both A_1 and A_2 make negative contributions to I_{yz}.

C.4 AREA MOMENTS OF INERTIA ABOUT INCLINED AXES; PRINCIPAL MOMENTS OF INERTIA

In some applications, especially in unsymmetric bending of beams (Section 6.6), it is necessary to determine the moments and products of inertia relative to inclined axes (y', z') when I_y, I_z, and I_{yz} are known. The *coordinate transformation* relating coordinates (y', z') to coordinates (y, z) can be deduced from Fig. C-10.

The angle θ is measured **positive counterclockwise from y to y'** (and z to z').

$$\begin{aligned} y' &= y\cos\theta + z\sin\theta \\ z' &= -y\sin\theta + z\cos\theta \end{aligned} \qquad \text{(C-20)}$$

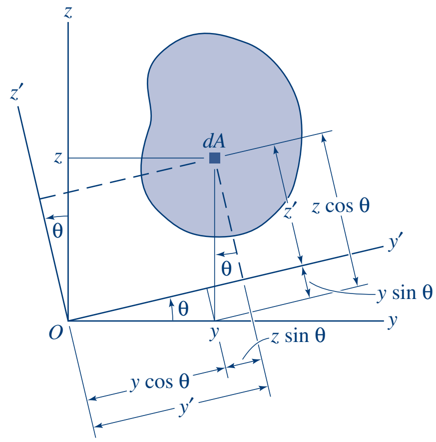


FIGURE C-10 Transformation of coordinates in a plane.

From Eqs. (C-5), (C-15), and (C-20),

$$I_{y'} = \int_A (z')^2\, dA = \int_A (-y \sin\theta + z\cos\theta)^2\, dA$$
$$I_{y'z'} = \int_A y'z'\, da = \int_a (y\cos\theta + z\sin\theta)(-y\sin\theta + z\cos\theta)\, dA \qquad \text{(C-21)}$$

Expanding each of the above integrands and recognizing that $\int_A y^2\, dA = I_z$, and so forth, we get

$$I_{y'} = I_y \cos^2\theta + I_z \sin^2\theta - 2I_{yz}\sin\theta\cos\theta$$
$$I_{y'z'} = (I_y - I_z)\sin\theta\cos\theta + I_{yz}(\cos^2\theta - \sin^2\theta)$$

These equations may be simplified by using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$. Thus,

$$I_{y'} = \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2}\cos 2\theta - I_{yz}\sin 2\theta$$
$$I_{y'z'} = \frac{I_y - I_z}{2}\sin 2\theta + I_{yz}\cos 2\theta \qquad \text{(C-22)}$$

Note the similarity between these equations and the stress-transformation equations, Eqs. 8-5.[6]

Principal Moments of Inertia. From Eqs. (C-22) it may be seen that $I_{y'}$ and $I_{y'z'}$ depend on the angle θ. We will now determine the orientations of the y' axis for which $I_{y'}$ takes on its maximum and minimum values. The axes having these orientations are called the *principal axes of inertia* of the area, and the corresponding moments of inertia are called the *principal moments of inertia*. To each point O in an area, there is a specific set of principal axes passing through that point. The principal axes that pass through the centroid of the area, called the *centroidal principal axes*, are the most important. The orientations of the centroidal principal axes for several unequal-leg angles are given in Appendix D.6.

The moment of inertia $I_{y'}$ will have a maximum, or minimum, value if the y' axis is oriented at an angle $\theta = \theta_p$ that satisfies the equation

$$\frac{dI_{y'}}{d\theta} = -2\left(\frac{I_y - I_z}{2}\right)\sin 2\theta - 2I_{yz}\cos 2\theta = 0$$

Therefore,

$$\tan 2\theta_p = \frac{-I_{yz}}{\left(\dfrac{I_y - I_z}{2}\right)} \qquad \text{(C-23)}$$

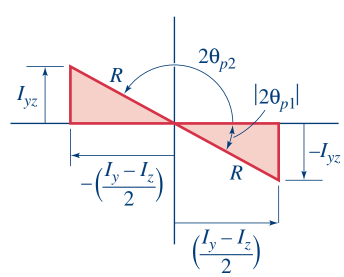


FIGURE C-11 Orientation of the principal axes of inertia.

Figure C-11 illustrates how to use the tangent value given by Eq. (C-23) to determine the angles θ_p. There are two distinct angles that satisfy Eq. (C-23). As

[6]There is a sign difference between the τ_{nt}-type terms and the I_{yz}-type terms, however.

illustrated by Fig. C-11, these two values of $2\theta_p$, labeled $2\theta_{p_1}$ and $2\theta_{p_2}$, differ by 180°, so the principal axes are oriented at 90° to each other (as they must be).

From Fig. C-11, the hypotenuse of either of the shaded triangles is given by

$$R = \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2} \tag{C-24}$$

Also, from Fig. C-11, the angles $2\theta_{p_1}$ and $2\theta_{p_2}$ satisfy

$$\sin 2\theta_{p_1} = \frac{-I_{yz}}{R}, \qquad \cos 2\theta_{p_1} = \frac{\left(\frac{I_y - I_z}{2}\right)}{R} \tag{C-25a}$$

$$\sin 2\theta_{p_2} = \frac{I_{yz}}{R}, \qquad \cos 2\theta_{p_2} = \frac{-\left(\frac{I_y - I_{yz}}{2}\right)}{R} \tag{C-25b}$$

Substituting these sines and cosines into the equation for $I_{y'}$, Eq. (C-22a), we get the following expressions for the two principal moments of inertia:

$$I_{\max} \equiv I_{p_1} = \frac{I_y + I_z}{2} + \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2}$$
$$I_{\min} \equiv I_{p_2} = \frac{I_y + I_z}{2} - \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2} \tag{C-26}$$

If Eqs. (C-25a) or Eqs. (C-25b) are substituted into Eq. (C-22b), it is found that

$$I_{p_1 p_2} = 0 \tag{C-27}$$

That is, the product of inertia with respect to the principal axes of inertia is equal to zero.

By adding Eqs. (C-26a) and (C-26b) we get

$$I_{p_1} + I_{p_2} = I_y + I_z \tag{C-28}$$

Thus, the sum of the moments of inertia about any pair of mutually perpendicular axes passing through a given point in a given plane is a constant.

EXAMPLE C-4

For the L-shaped area in Fig. 1 of Example C-2, (a) Determine the orientation of the centroidal principal axes and show the orientation on a sketch. (b) Determine the principal moments of inertia.

$$I_y = \frac{5894}{147}t^4 = 40.10t^4, \quad I_z = \frac{33{,}103}{294}t^4 = 112.60t^4$$

$$I_{yz} = \frac{-270}{7}t^4 = -38.57t^4$$

Solution (a) From Eq. (C-23),

$$\tan 2\theta_p = \frac{-I_{yz}}{\left(\dfrac{I_y - I_z}{2}\right)} = \frac{-\left(\dfrac{-270}{7}\right)}{\dfrac{11{,}788 - 33{,}103}{2(294)}} = -1.064$$

$$2\theta_{p_1} = 133.22°, \quad 2\theta_{p_2} = -46.78°$$

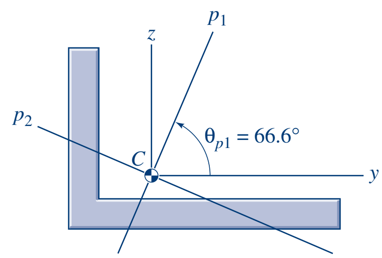


Fig. 1

Then, as illustrated in Fig. 1,

$$\theta_{p_1} = 66.6°, \quad \theta_{p_2} = -23.4° \qquad \textbf{Ans.}$$

(b) From Eq. (C-26a),

$$I_{p_1} = \frac{I_y + I_z}{2} + \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2}$$

$$= \frac{40.10t^4 + 112.60t^4}{2} + \sqrt{\left(\frac{40.10t^4 - 112.60t^4}{2}\right)^2 + (-38.57t^4)^2}$$

$$= 129.28t^4$$

or

$$I_{p_1} = 129.3t^4 \qquad \textbf{Ans.}$$

Similarly, from Eq. (C-26b),

$$I_{p_2} = 23.4t^4 \qquad \textbf{Ans.}$$

Mohr's Circle for Moments and Products of Inertia. Equations (C-22) have the same basic form as Eqs. 8.5, which were used to develop Mohr's circle for stress.[7] Therefore, by a procedure that is virtually identical to that in Section 8.5, it can be shown that a Mohr's circle plotted as in Fig. C-12 can be used to

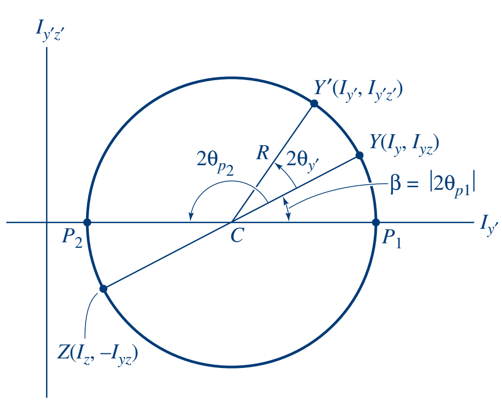


FIGURE C-12 Mohr's circle for moments and products of inertia.

[7]There is a difference between the signs preceding the τ_{xy}-type terms in Eqs. 8.5 and the signs preceding the corresponding I_{yz}-type terms in Eqs. (C-22). Thus, for Mohr's circle for moments and products of inertia, the $I_{y'z'}$ axis is positive upward, not positive downward, as it was for Mohr's circle for stress.

compute $I_{y'}$ and $I_{y'z'}$ for any (y', z') axes located at angle θ counterclockwise from the given (y, z) axes. And the Mohr's circle provides a convenient way to calculate the orientation of the principal axes of inertia and the principal moments of inertia, I_{p_1} and I_{p_2}, given moments of inertia I_y and I_z and the corresponding product of inertia I_{yz}. To an angle θ measured counterclockwise (or clockwise) on the planar area A, there corresponds an angle 2θ measured counterclockwise (or clockwise) on Mohr's circle.

The following procedure will facilitate your calculation of moments and products of inertia with respect to rotated axes.

MOHR'S-CIRCLE PROCEDURE FOR MOMENTS AND PRODUCTS OF INERTIA

1. Establish a set of Mohr's-circle axes $(I_{y'}, I_{y'z'})$, as shown in Fig. C-12. (Note that the positive $I_{y'z'}$ axis is counterclockwise 90° from the $I_{y'}$ axis, unlike the τ_{nt} axis for Mohr's circle of stress in Chapter 8.)
2. Plot points $Y{:}(I_y, +I_{yz})$ and $Z{:}(I_z, -I_{yz})$, respectively.
3. Draw a straight line joining points Y and Z. The intersection of the YZ line with the $I_{y'}$ axis is the center of the Mohr's circle passing through points Y and Z.
4. Point Y', located at angle $2\theta_{y'}$ counterclockwise from the line CY, as shown in Fig. C-12, locates the point whose coordinates are $(I_{y'}, +I_{y'z'})$.
5. Points P_1 and P_2 locate the two principal axes at $2\theta_{p_1}$ and $2\theta_{p_2}$, respectively, as shown in Fig. C-12. The principal moments of inertia are I_{p_1} and I_{p_2}, which are also given by Eqs. (C-26).

EXAMPLE C-5

(a) Draw the Mohr's circle for the centroidal moments and products of inertia for the L-shaped area in Fig. 1 of Example C-2, given that:

$$I_y = \frac{5894}{147}t^4 = 40.10t^4, \quad I_z = \frac{33{,}103}{294}t^4 = 112.60t^4$$

$$I_{yz} = \frac{-270}{7}t^4 = -38.57t^4$$

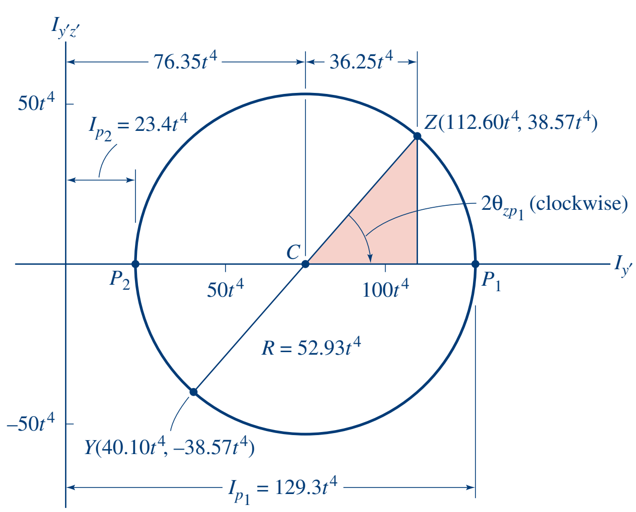


Fig. 1 Mohr's circle for centroidal inertias of an L-shaped area.

(b) Use the Mohr's circle constructed in Part (a) to compute the principal moments of inertia I_{p_1} and I_{p_2} and to locate the principal axes. Show the orientation of the principal axes on a sketch.

Solution (a) Sketch Mohr's circle and calculate the principal moments of inertia. Points Y and Z are plotted and Mohr's circle is then drawn (Fig. 1). From the circle,

$$I_{\text{avg.}} = \frac{40.10t^4 + 112.60t^4}{2} = 76.35t^4$$

$$R = \sqrt{\left(\frac{112.60t^4 - 40.10t^4}{2}\right)^2 + (38.57t^4)^2} = 52.93t^4$$

$$I_{p_1} = I_{\text{avg.}} + R = 129.3t^4$$

$$I_{p_2} = I_{\text{avg.}} - R = 23.4t^4$$

Ans. (a)

(b) Determine the orientation of the principal axes and show them on a sketch.

$$\tan|2\theta_{zp_1}| = \frac{38.57t^4}{\left(\dfrac{112.60t^4 - 40.10t^4}{2}\right)} = 1.064$$

Therefore, $2\theta_{zp_1} = 46.78°$ (clockwise), so

$$\theta_{zp_1} = \theta_{yp_2} = 23.4° \text{ clockwise}$$

Ans. (b)

Note that the orientations of the principal axes in Fig. 2 are such that the contributions to $I_{p_1p_2}$ of the areas in the four quadrants cancel out, giving $I_{p_1p_2} = 0$.

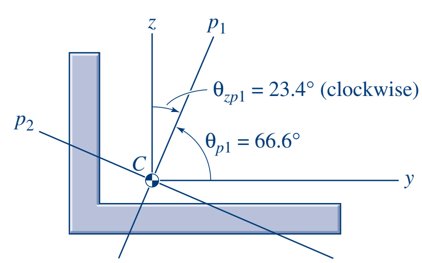


Fig. 2 Principal axes of inertia.

The results obtained from Mohr's circle are the same as those obtained by the use of formulas in Example C-4. However, mistakes are less likely to be made if Mohr's circle is carefully drawn and it is recalled that an angle 2θ on Mohr's circle corresponds to an angle θ on the planar area A, and that angles are taken in the same sense on Mohr's circle as on the planar area.

MDS6.1 **Beam Cross-Sectional Properties—***Section Properties* is an MDS computer program module for calculating section properties of plane areas: area, location of centroid, moments of inertia, product of inertia, orientation of principal axes, etc., properties that are defined and illustrated in **Appendix C.** The *Section Properties* module is closely linked with the *Flexure* module.